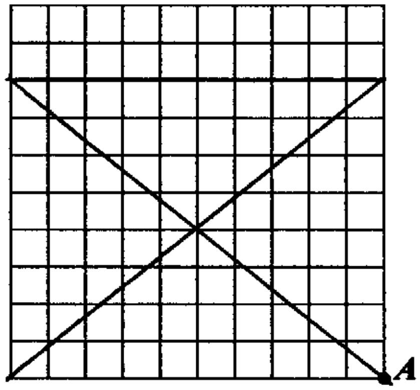
# *2013 New Hampshire State Mathematics Contest*

**Sponsored by USNH and New Hampshire Teachers of Mathematics**

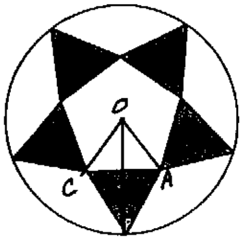
Team (category 1)
(35 minutes)

1. Starting at point $A$, draw three connected line segments, which are entirely within or on the grid, that separate the whole grid into five regions of equal area.




Answer____________________

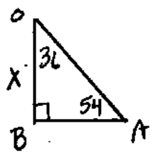
2. In the figure, all shaded triangles are equilateral. The length of a side of the regular pentagon is 1 inch. Find the radius of the circle.




In equilateral triangle, height $= \frac{\sqrt{3}}{2}$

In regular pentagon, $m\angle AOC = \frac{360}{5} = 72°$, then $m\angle AOB = 36°$.

Measure of an interior angle $= 180 - \frac{360}{5} = 108°$, then $m\angle OAB = 54°$




$$\frac{\sin 54°}{X} = \frac{\sin 36°}{.5}$$

$$X = \frac{.5(\sin 54°)}{\sin 36°} = .6881909602$$

$$\text{Radius} = \frac{\sqrt{3}}{2} + .6881909602 \doteq 1.554$$

Answer 1.554 in

3. Find 20 consecutive natural numbers, none of which is a prime number.

Consider $n+2$ where $n$ has a factor of 2, then $n+2$ not a prime.

Similarly, $n+3$ where $n$ has a factor of 3 is not a prime,

$n+4$ where $n$ has a factor of 4 is not a prime, ..., etc.

Need 20 such non-prime numbers, then 20th number is $n+21$.

Since $n$ must have factors of $2, 3, 4, \ldots, 21$ then $n$ could be $21!$

One possible solution: $21!+2, 21!+3, 21!+4, \ldots, 21!+21$

Answer $21!+2, 21!+3, 21!+4, \ldots, 21!+21$

* OTHER SOLUTIONS EXIST

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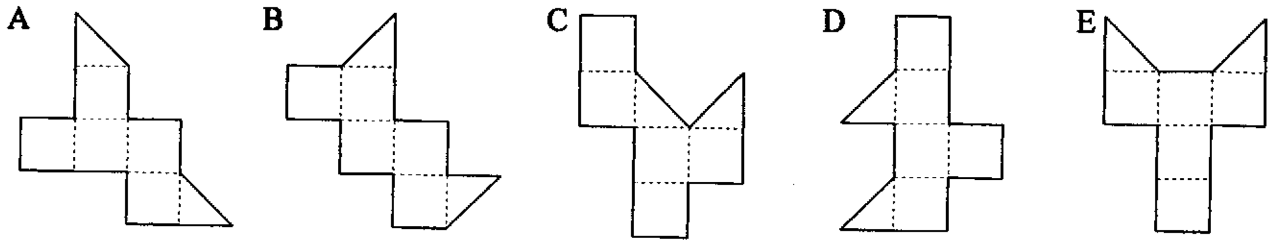
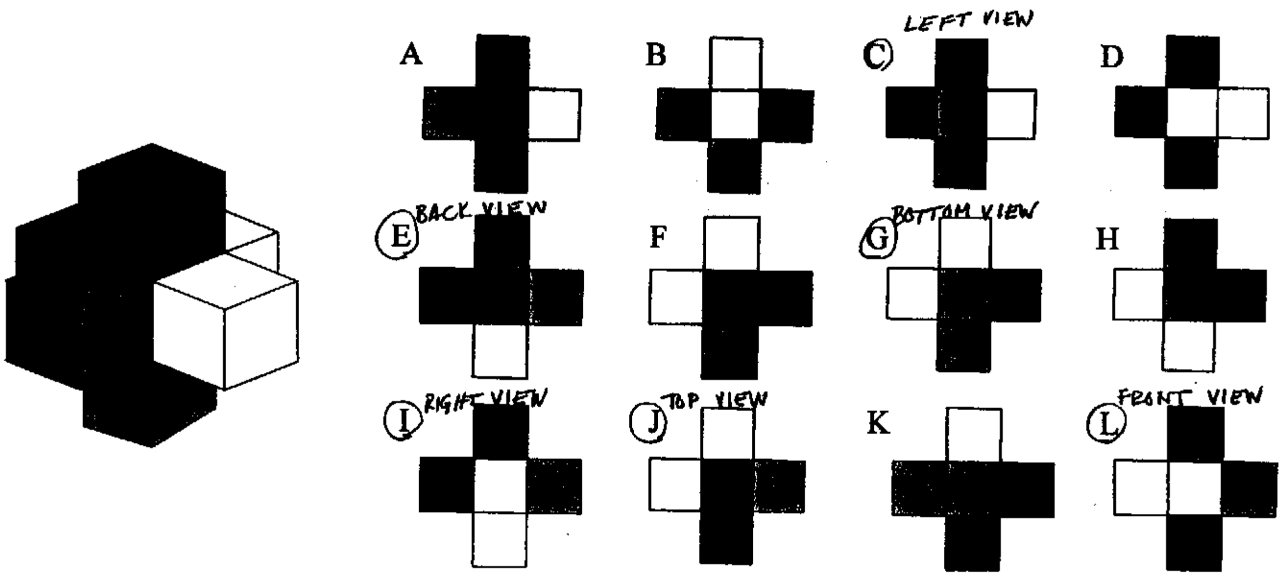
Recreational Math (category 2)
(12 minutes)

1. Which patterns below form a closed cube when folded along the dotted lines? List all possibilities.



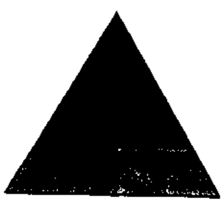

Answer C + D

2. The 3-D shape consists of six cubes in three shades. Twelve 2-D views (A-L) are shown. Which are correct? List all possibilities.




Answer C, E, G, I, J, L

3. An equilateral triangle is divided into two shapes: a small triangle and a trapezoid. Each shape has an area of 1. If the ratio of the height of the small triangle to the height of the trapezoid is $1:x$, find the exact value of $x$.



$$\frac{\text{Area small } \Delta}{\text{Area large } \Delta} = \frac{1}{2} \quad \text{then} \quad \frac{h_1}{h_1+h_2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$h_1 = 1$

$1 + h_2 = \sqrt{2} \Rightarrow h_2 = \sqrt{2} - 1$

Answer $\sqrt{2} - 1$

Source: *The Big, Big, Big Book of Brainteasers*, The Grabarchuk Family, Puzzle Wright Press, New York, 2011

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## Algebra I (category 3)

(12 minutes)

1. How many integers between 10,001 and 20,000 are perfect squares?

$10\,001 < x^2 < 20\,000$

$100.005 < x < 141.42$

$x \in \{101, 102, 103, \ldots, 141\}$

41 integers

Answer 41

2. What is the value of $x$ if $6^{x+1} - 6^x = 1080$?

$6^x(6^1 - 1) = 1080$

$6^x = 216$

$x = 3$

Answer 3

3. Find the product $pq$ such that each solution of the equation $x^2 + px + q = 0$ is one less than a solution of the equation $x^2 - 5x - 6 = 0$.

$(x-6)(x+1) = 0$

$x = 6 \quad x = -1$

$6 - 1 = 5 \quad -1 - 1 = -2$

$(x-5)(x+2) = 0$

$x^2 - 3x - 10 = 0$

$pq = (-3)(-10) = 30$

Answer 30

## *2013 New Hampshire State Mathematics Contest*

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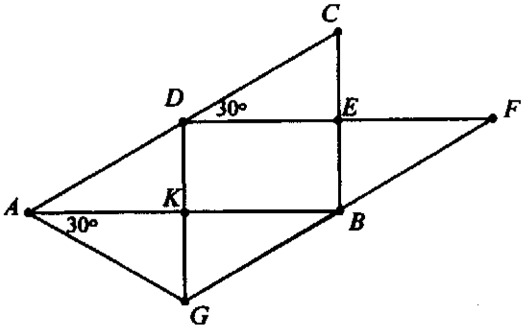
### Geometry (category 4)
(12 minutes)

1. *DEBK* is a rectangle and the length of $\overline{BE}$ is 1 cm. What is the length of $\overline{AG}$?




$m\angle DAK = 30°$ by transversal of parallel lines

$\triangle DAK \cong \triangle GAK$ by ASA

$EB = DK = GK = 1$

$AG = 2$ by $30°$-$60°$-$90°$ triangle

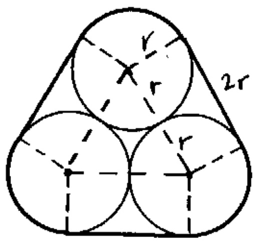
Answer ___2___ cm

2. The length of a band wrapped tightly around three coplanar circular disks is 12 centimeters. If the disks have equal radii and are tangent to each other as shown, what is the radius of each disk in centimeters?

Let $r$ be the radius of all circles.

the three arcs between the tangent segments form a circle with radius $r$.

Each tangent segment has length $2r$.



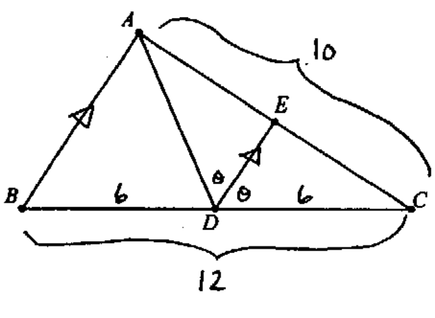

$12 = 3(2r) + 2\pi r$

$12 = r(6 + 2\pi)$

$r = \frac{12}{6+2\pi} = \frac{6}{3+\pi} \doteq .977$

Answer ___$\frac{6}{3+\pi}$ OR $.977$___

3. In $\triangle ABC$, $BC = 12$, $AC = 10$, $D$ is the midpoint of $\overline{BC}$ and $E$ lies on $\overline{AC}$ so that $m\angle ADE = m\angle CDE$ and $\overline{DE}$ is parallel to $\overline{AB}$. What is the length of $\overline{ED}$?




Let $m\angle ADE = m\angle CDE = \theta$

$\left.\begin{array}{l} m\angle ABD = \theta \\ m\angle BAD = \theta \end{array}\right\}$ transversal of parallel lines

$\triangle ABD$ is isosceles then $BD = AD = 6$

$\triangle ADE \cong \triangle CDE$ by SAS

$m\angle AED = m\angle CED = 90$

$AE = CE = 5$

$\triangle CDE$ is a right triangle

Answer ___$\sqrt{11}$ OR $3.317$___ units

$ED = \sqrt{6^2 - 5^2} = \sqrt{11} \doteq 3.317$

# *2013 New Hampshire State Mathematics Contest*

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## Algebra II (category 5)
## (12 minutes)

1. Suppose that $\frac{x}{y}=\frac{4}{7}$ and $\frac{y}{z}=\frac{14}{3}$. What is the value of $\frac{x+y}{z}$?

$$\frac{x}{y}=\frac{4}{7} \Rightarrow x=\frac{4y}{7}$$

$$\frac{y}{z}=\frac{14}{3} \Rightarrow z=\frac{3y}{14}$$

$$\frac{x+y}{z}=\frac{\frac{4y}{7}+y}{\frac{3y}{14}}=\frac{\frac{11y}{7}}{\frac{3y}{14}}=\frac{11y}{7}\cdot\frac{14}{3y}=\frac{22}{3}$$

Answer $\frac{22}{3}$ OR $7\frac{1}{3}$ OR $7.333$

2. Let $P(x)=kx^3+2k^2x^2+k^3$. Find the sum of all real numbers $k$ for which $x-2$ is a factor of $P(x)$.

$$P(2)=k(2^3)+2k^2(2^2)+k^3$$
$$=8k+8k^2+k^3$$
$$=k(8+8k+k^2)$$

$\uparrow$

Sum of roots $=-8$

Answer $-8$

3. Find the limiting value of the following continued fraction.

$$\text{LET } x= 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}$$

then $x=1+\frac{1}{x}$

$$x^2=x+1$$
$$x^2-x-1=0$$
$$x=\frac{1\pm\sqrt{1+4}}{2}=\frac{1\pm\sqrt{5}}{2}$$

Since fraction is positive $x=\frac{1+\sqrt{5}}{2}\doteq 1.618$

Answer $\frac{1+\sqrt{5}}{2}$ OR $1.618$

## *2013 New Hampshire State Mathematics Contest*

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### Advanced Math (category 6)

(12 minutes)

1. Suppose the function $f$ is defined as: $f(n) =$ the $n^{\text{th}}$ digit to the right of the decimal point in the decimal expansion of $\frac{1}{7}$. What is the value of $f(2013)$?

$\frac{1}{7} = .\overline{142857}$

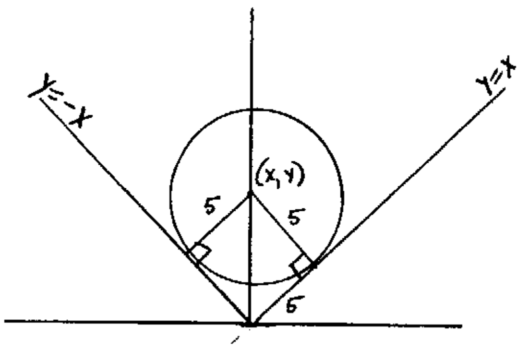
Expansion repeats every 6 digits

$\frac{2013}{6} = 335 \text{ r } 3$

Third digit in pattern is 2

Answer 2

2. Find an equation for the circle with radius 5 which is tangent to both branches of the graph of $y = |x|$.




$y = |x|$ then $y = x$ or $y = -x$, $y \geq 0$

Y coordinate of center is the hypotenuse of a 45°-45°-90° triangle then $y = 5\sqrt{2}$ and $x = 0$

$x^2 + (y - 5\sqrt{2})^2 = 25$

$x^2 + y^2 - 10\sqrt{2}y + 50 = 25$

$x^2 + y^2 - 10\sqrt{2}y + 25 = 0$

Answer $x^2 + (y - 5\sqrt{2})^2 = 25$ OR $x^2 + y^2 - 10\sqrt{2}y + 25 = 0$

3. Find all values of $x$ for which $\left(\sqrt{2}\right)^{\tan x} = \left(\frac{1}{2}\right)^{\sin x}$. Give exact values.

$(\sqrt{2})^{\frac{\sin x}{\cos x}} = \left(\frac{1}{2}\right)^{\sin x}$

when $\sin x = 0$, $x \in \{\pi k$ where $k$ is an integer$\}$

$\log (\sqrt{2})^{\frac{\sin x}{\cos x}} = \log \left(\frac{1}{2}\right)^{\sin x}$

$\frac{\sin x}{\cos x} \log(\sqrt{2}) = \sin x \log\left(\frac{1}{2}\right)$

$\frac{1}{\cos x} \log(\sqrt{2}) = \log\left(\frac{1}{2}\right)$

$\frac{1}{\cos x}\left[\frac{1}{2} \log 2\right] = -1 \log 2$

$\cos x = -\frac{1}{2}$ $\quad x \in \{\frac{2}{3}\pi + 2\pi k$, $k$ is an integer$\}$

$x \in \{\frac{4}{3}\pi + 2\pi k$, $k$ is an integer$\}$

Answer $\{\pi k, k \in \mathbb{Z}\} \cup \{\frac{2}{3}\pi + 2\pi k, k \in \mathbb{Z}\} \cup \{\frac{4}{3}\pi + 2\pi k, k \in$